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GROUP CLASSIFICATION OF THE TWO-DIMENSIONAL EQUATIONS OF MOTION OF A VISCOUS HEAT-CONDUCTING PERFECT GAS

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1. General Statement of the Problem. The group properties of a system of differential equations which describe plane ($\nu = 0$) and axisymmetrical ($\nu = 1$) motions of a viscous heat-conducting perfect gas are investigated:

$$\rho_t + u\rho_x + v\rho_y + \rho u_x + \rho v_y + \nu\frac{\rho u}{x} = 0; \tag{1.1}$$

$$\rho(u_t + uu_x + vu_y) = -p_x - \frac{2}{3}\Big(\mu\Big(u_x + v_y + \nu\frac{u}{x}\Big)\Big)_x + 2(\mu u_x)_x + (\mu(u_y + v_x))_y + 2\nu\mu\Big(\frac{u}{x}\Big)_x; \tag{1.2}$$

$$\rho(v_t + uv_x + vv_y) = -p_y - \frac{2}{3}\Big(\mu\Big(u_x + v_y + \nu\frac{u}{x}\Big)\Big)_y + 2(\mu u_y)_y + (\mu(u_y + v_x))_x + \nu\frac{\mu}{x}(u_y + v_x); \tag{1.3}$$

$$p_t + up_x + vp_y + p\Big(1 + \frac{R}{\varepsilon'}\Big)\Big(u_x + v_y + \nu\frac{u}{x}\Big) = \frac{1}{\varepsilon'}\bigg(\Big(æ\Big(\frac{p}{\rho}\Big)_x\Big)_x + \Big(æ\Big(\frac{p}{\rho}\Big)_y\Big)_y + \nu\frac{æ}{x}\Big(\frac{p}{\rho}\Big)_x\bigg) +$$
$$+2\mu\frac{R}{\varepsilon'}\Big(\frac{2}{3}\Big(u_x^2 - u_xv_y + v_y^2 + \nu\frac{u}{x}\Big(\frac{u}{x} - u_x - v_y\Big)\Big) + \frac{1}{2}(v_x + u_y)^2\Big). \tag{1.4}$$

Here ρ is the density, p is the pressure, μ is the viscosity coefficient, $æ$ is the coefficient of heat conductivity, ε is the internal energy of the gas, $\varepsilon' = d\varepsilon/dT$, and T is the temperature.

The gas obeys the Clapeyron equation $p = R\rho T$. The internal energy of a perfect gas depends only on temperature [1]. The coefficients of viscosity and of heat conductivity are also considered dependent only on temperature:

$$\mu = \mu(p/\rho), \quad æ = æ(p/\rho), \quad \varepsilon = \varepsilon(p/\rho). \tag{1.5}$$

For Eqs. (1.1)–(1.4), the problem of group classification with respect to the arbitrary elements μ, $æ$, and ε [2] is posed. The gas is considered essentially viscous and heat-conducting ($\mu \neq 0$ and $æ \neq 0$).

The full group of transformations is sought in the space of variables t, x, y, u, v, ρ, and p.

2. Equivalence Transformation. Group classification requires finding the equivalence group admitted by Eqs. (1.1)–(1.4). The following designations will be used everywhere:

$$q = (t, x, y), \quad w = (u, v, \rho, p), \quad \tau = (\varepsilon', \mu, æ, \mu', æ'), \quad h = (t, x, y, u, v, \rho, p),$$
$$w_i^k = \partial w^k/\partial q^i, \quad w_{ij}^k = \partial w_i^k/\partial q^j, \quad \tau_m^l = \partial\tau^l/\partial h^m$$
$$(k = 1, \ldots, 4;\ i, j = 1, 2, 3;\ l = 1, \ldots, 5;\ m = 1, \ldots, 7).$$

To system (1.1)–(1.4) we add the following conditions on the functions μ, $æ$, and ε', which are a consequence of the assumptions (1.5):

$$\tau_i^k = 0 \quad (k, i = 1, \ldots, 5); \tag{2.1}$$

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$$\frac{\partial \mu}{\partial p} = \frac{\mu'}{R\rho}, \qquad \frac{\partial æ}{\partial p} = \frac{æ'}{R\rho}, \qquad \frac{\partial \varepsilon'}{\partial p} = \frac{\varepsilon''}{R\rho}; \tag{2.2}$$

$$\frac{\partial \mu}{\partial \rho} = -\frac{p\mu'}{R\rho^2}, \qquad \frac{\partial æ}{\partial \rho} = -\frac{pæ'}{R\rho^2}, \qquad \frac{\partial \varepsilon'}{\partial \rho} = -\frac{p\varepsilon''}{R\rho^2}. \tag{2.3}$$

In system (1.1)–(1.4), the derivatives of viscosity and heat conductivity with respect to space are written in the form

$$\frac{\partial \mu}{\partial x} = \left(\frac{p_x}{\rho} - \frac{p\rho_x}{\rho^2}\right)\mu', \quad \frac{\partial \mu}{\partial y} = \left(\frac{p_y}{\rho} - \frac{p\rho_y}{\rho^2}\right)\mu', \qquad \frac{\partial æ}{\partial x} = \left(\frac{p_x}{\rho} - \frac{p\rho_x}{\rho^2}\right)æ',$$

$$\frac{\partial æ}{\partial y} = \left(\frac{p_y}{\rho} - \frac{p\rho_y}{\rho^2}\right)æ', \qquad \mu' = d\mu/dT, \qquad æ' = dæ/dT, \qquad \varepsilon'' = d\varepsilon'/dT.$$

For system (1.1)–(1.4), (2.1)–(2.3), we solve the problem of constructing the full group of transformations of the space of variables t, x, y, u, v, ρ, p, μ, $æ$, ε', μ', and $æ'$ with the admissible operator $X^e = \xi^i\partial_{q^i} + \eta^j\partial_{w^j} + \alpha^k\partial_{\tau^k}$, where ξ^i, η^j, and α^k are functions of t, x, y, u, v, ρ, p, μ, $æ$, ε', μ', and $æ'$.

The coordinates of the prolonged operator $X^e_p = X^e + \zeta^i_j\partial_{w^i_j} + \zeta^i_{jk}\partial_{w^i_{jk}} + \beta^i_j\partial_{\tau^i_j}$, with allowance for (2.1), are found from the formulas

$$\zeta^i_j = D_j\eta^i - w^i_k D_j\xi^k, \qquad \zeta^i_{jk} = D_k\zeta^i_j - w^i_{jl}D_k\xi^l, \qquad \beta^i_j = D'_j\alpha^i - \tau^i_6 D'_j\eta^3 - \tau^i_7 D'_j\eta^4.$$

Here

$$D_j = \partial_{q^j} + w^k_j\partial_{w^k} + w^k_{ij}\partial_{w^k_i} \quad (j = 1, 2, 3);$$

$$D'_j = \partial_{h^j} \quad (j = 1, \ldots, 5); \quad D'_k = \partial_{h^k} + \tau^i_k\partial_{\tau^i} \quad (k = 6, 7).$$

The difference in the calculation of the coordinates ζ^i_j and β^i_j arises from the fact that the variables u, v, ρ, and p and ε', μ, $æ$, μ', and $æ'$ are present in different spaces.

Computation of the equivalence group leads to transformations that correspond to the operator

$$t\partial_t + x\partial_x + y\partial_y + \mu\partial_\mu + æ\partial_æ + \mu'\partial_{\mu'} + æ'\partial_{æ'} \tag{2.4}$$

and to the kernel of the full groups ($\nu = 0, 1$), which is written below.

3. Result of the Group Classification. Equations (1.1)–(1.4) are treated as a system of second-order differential equations for four unknown functions: u, v, ρ, and p. The operator admitted by these equations is sought for in the form

$$X = \xi^i\partial_{q^i} + \eta^j\partial_{w^j},$$

where ξ^i, and η^j are functions of t, x, y, u, v, ρ, and p.

The algorithm for finding the full group [2] requires a large body of intermediate calculations. Thus, the number of defining equations in this case equals 28192, and therefore they were derived on a computer. The "Reduce" algebraic programming system [3] was written for these computations.

Computations lead to the classifying equations

$$c\left(R\frac{\mu}{\mu'} - \frac{p}{\rho^2}\left(\frac{\mu}{\mu'}\right)'\right) = 0, \quad c\left(R\frac{æ}{æ'} - \frac{p}{\rho^2}\left(\frac{æ}{æ'}\right)'\right) = 0; \tag{3.1}$$

$$c(\mu/æ)' = 0, \qquad c\varepsilon'' = 0 \tag{3.2}$$

with the constant c related to the coordinates ξ^i and η^k of the infinitesimal operator:

$$\xi^1 = c_6 + c_7 t, \quad \xi^2 = \nu(c_1 + c_3 t + c_5 y) + (c_7 + c)x, \quad \xi^3 = c_2 + c_4 t - \nu c_5 x + (c_7 + c)y,$$

$$\eta^1 = \nu(c_3 + c_5 v) + cu, \quad \eta^2 = c_4 - \nu c_5 u + cv, \quad \eta^3 = (-c_7 + 2(\omega - 1)c)\,\rho, \quad \eta^4 = (-c_7 + 2\omega c)p.$$

In the determination of the kernel of the full groups it is assumed that the functions μ, $æ$, and ε are

TABLE 1

No.	ε	μ	æ	r	Basis L_r
1	$f\left(\frac{p}{\rho}\right)$	$g\left(\frac{p}{\rho}\right)$	$h\left(\frac{p}{\rho}\right)$	7	1, 2, 3, 4, 5, 6, 7
2	$\frac{p}{(\gamma-1)\rho}$	$\left(\frac{p}{\rho}\right)^{\omega}$	$æ_0\left(\frac{p}{\rho}\right)^{\omega}$	8	1, 2, 3, 4, 5, 6, 7, 8

TABLE 2

	X_2	X_4	X_6	X_7	X_8
X_2	0	0	0	X_2	X_2
X_4	0	0	$-X_2$	0	X_4
X_6	0	X_2	0	X_6	0
X_7	$-X_2$	0	$-X_6$	0	0
X_8	$-X_2$	$-X_4$	0	0	0

arbitrary and, hence, $c = 0$. Therefore, in the case of plane symmetry ($\nu = 0$), the kernel is the seven-parameter group and, in the case of axial symmetry ($\nu = 1$), the kernel is the four-paramer group.

The specialization of the elements μ, æ, and ε follows from Eqs. (3.1) and (3.2).

Table 1 gives the result of group classification for the case of plane symmetry, accurate to the equivalence transformations (2.4). Here f, g, and h are arbitrary functions of specified arguments and $\gamma \neq 1$, ω, and $æ_0$ are arbitrary constants. The basis of the principal Lie algebra is represented by the numbers of the operators

$$X_1 = \partial_x,\ X_2 = \partial_y,\ X_3 = t\partial_x + \partial_u,\ X_4 = t\partial_y + \partial_v,\quad X_5 = y\partial_x - x\partial_y + v\partial_u - u\partial_v,\ X_6 = \partial_t,$$

$$X_7 = t\partial_t + x\partial_x + y\partial_y - \rho\partial_\rho - p\partial_p,\quad X_8 = x\partial_x + y\partial_y + u\partial_u + v\partial_v + 2(\omega-1)\rho\partial_\rho + 2\omega p\partial_p.$$

In the case of axial symmetry, the equations do not admit the operators X_1, X_3, and X_5 and in other respects the result of the classification coincides with the result given in Table 1.

4. Exact Solutions. For a model of a polytropic gas with an exponential temperature dependence of viscosity and of heat conductivity, invariant and partially invariant solutions were constructed. In this work only some solutions to the equations of axisymmetric motion of the gas are given.

To distinguish classes of essentially dissimlar solutions, it is required to construct an optimal system of subalgebras of the Lie algebra $L_5 = \{X_2, X_4, X_6, X_7, X_8\}$. For this purpose, the algebraic approach to constructing the optimal system of subalgebras is used, which was recently developed in [4–6].

Table 2 is the commutator table for the algebra Lie L_5.

Table 3 gives actions of the internal automorphisms of the Lie algebra L_5 on the coordinates of the vector $X = x^i X_i$, where two discrete automorphisms, E_1 and E_2, which correspond to reversal of the directions of the y and t axes (in the first case) and of the y axis (in the second case), are added.

Table 4 shows the optimal system of subalgebras of the algebra Lie L_5. The first integer in the number of the subalgebra is its dimension, and the second figure is its number among subalgebras of the given dimension. The coefficient α can acquire any real value, and the coefficients β and γ cannot acquire the values 0 and -1, respectively. Self-normalized subalgebras are marked by the equality sign. The superscript means that in the indicated subalgebra the value of the parameter is taken equal to the superscript (for example, 4.1^0 designates the subalgebra 4.1 with $\alpha = 0$).

As examples of exact solutions we consider the solutions obtained on the basis of three-dimensional subalgebras. In this case invariant solutions are represented as a system of algebraic equations, and partially invariant solutions are represented as a system of ordinary differential equations.

TABLE 3

Automorphism	x^2	x^4	x^6	x^7	x^8
A_2	$x^2 + a_2(x^7 + x^8)$	x^4	x^6	x^7	x^8
A_4	$x^2 - a_4 x^4$	$x^4 + a_4 x^8$	x^6	x^7	x^8
A_6	$x^2 + a_6 x^4$	x^4	$x^6 + a_6 x^7$	x^7	x^8
A_7	$a_7 x^2$	x^4	$a_7 x^6$	x^7	x^8
A_8	$a_8 x^2$	$a_8 x^4$	x^6	x^7	x^8
E_1	$-x^2$	x^4	$-x^6$	x^7	x^8
E_2	x^2	$-x^4$	$-x^6$	x^7	x^8

By a physical solution is meant a solution of system (1.1)–(1.4) that satisfies the conditions $\rho > 0$ and $p > 0$.

The subalgebras 3.1 ($\alpha \neq -1$), 3.4, 3.5, 3.6, 3.7, and 3.9 satisfy the necessary condition of existence of an invariant solution [2, Theorem 19.3]. The solutions constructed on the basis of these subalgebras are given below.

Submodel 3.1 ($\alpha \neq -1$). We consider the following cases:
(a) for $\alpha = 0$ the solution has the form $u \equiv u_0 \neq 0$, $v = 0$, $\rho = \rho_0/x$, $p = p_0/x$, and the constants u_0, ρ_0, and p_0 are related by the relation

$$\left(\frac{p_0}{\rho_0}\right)^{\omega} = \frac{3}{4}\frac{p_0}{u_0};$$

(b) for $\alpha = -1/2$, the solution is the rest:

$$u = v = 0, \quad \rho = \rho_0 x^{-2\omega}, \quad p = p_0 x^{-2(\omega+1)};$$

(c) for other α ($\alpha \neq 0$ and $\alpha \neq -1/2$), physical solutions exist only for $\omega = 0$, i.e., when μ and $æ$ are constant

$$u = u_0 x^{\alpha/(\alpha+1)}, \quad v = v_0 x^{\alpha/(\alpha+1)}, \quad \rho = \rho_0 x^{-(2\alpha+1)/(\alpha+1)}, \quad p = p_0 x^{-1/(\alpha+1)}.$$

The constants u_0, v_0, ρ_0, and p_0 can be related in two ways:

(1)
$$v_0 = 0, \qquad p_0 = \left(\alpha\rho_0 u_0 + \frac{4}{3}\frac{2\alpha+1}{\alpha+1}\right)u_0,$$

$$(1 - 2\alpha\gamma - \gamma)\rho_0^2 u_0^2 + \left(4\frac{\alpha^2}{\alpha+1}\frac{\gamma-1}{R}æ_0 - \frac{4}{3}\frac{\alpha-1}{\alpha+1} - 4\frac{(\alpha-1)^2}{(\alpha+1)^2}\gamma\right)\rho_0 u_0 + \frac{16}{3}\frac{2\alpha+1}{(\alpha+1)^2}\frac{\gamma-1}{R}æ_0 = 0;$$

(2)
$$p_0 = \frac{\alpha(3\alpha^2 - 8\alpha + 4)}{3(\alpha+1)^2}\frac{1}{\rho_0},$$

$$\alpha^2(\gamma-1)v_0^2 + \frac{\alpha^3}{(\alpha+1)^3}\frac{1}{\rho_0^2}\left(\frac{4}{3} - 4\gamma + \frac{4}{3}(3\alpha^2 - 8\alpha + 4)\frac{\gamma-1}{R}æ_0 - 2\alpha^2\gamma - 5\alpha\gamma - \frac{\alpha}{3}\right) = 0.$$

Submodel 3.4. Physical solutions exist only for $\omega \neq 0$:

$$u = \frac{\omega-1}{\omega}\frac{x}{t}, \qquad v = v_0\frac{x}{t} + \frac{y}{t}, \qquad \rho = \rho_0 x^{2(\omega-1)} t^{1-2\omega},$$

$$p = p_0 x^{2\omega} t^{-2\omega-1}, \qquad \left(\frac{p_0}{\rho_0}\right)^{\omega} = \frac{3}{4}\frac{\omega-1}{\omega^2}\rho_0 - \frac{3}{2}\omega p_0.$$

TABLE 4

Number of Subalgebra	Basis	Normalizer
4.1	2, 4, 6, $7+\alpha 8$	L_5
4.2	2, 6, 7, 8	= 4.2
4.3	2, 4, 7, 8	= 4.3
4.4	2, 4, 6, 8	L_5
3.1	2, 6, $7+\alpha 8$	4.2
3.2	2, 4, $7+\alpha 8$	4.3
3.3	6, 7, 8	= 3.3
3.4	4, 7, 8	= 3.4
3.5	2, 7, 8	= 3.5
3.6	2, $4+6$, $7+8$	= 3.6
3.7	2, 6, $4+7$	4.1^0
3.8	2, 4, $6+8$	4.4
3.9	2, 6, 8	4.2
3.10	2, 4, 8	4.3
3.11	2, 4, 6	L_5
2.1	6, $7+\alpha 8$	3.3
2.2	4, $7+\alpha 8$	4.4
2.3	2, $7+\beta 8$	3.5
2.4	7, 8	= 2.4
2.5	6, $2+7-8$	3.1^{-1}
2.6	4, $2+7-8$	3.2^{-1}
2.7	4, $7-8$	4.3
2.8	$4+6$, $7+8$	= 2.8
2.9	2, $4+7$	3.2^0
2.10	2, 7	4.3
2.11	2, $6+8$	3.9
2.12	6, 8	4.2
2.13	4, 8	3.4
2.14	2, 8	4.2
2.15	2, $4+6$	3.11
2.16	2, 6	L_5
2.17	2, 4	L_5
1.1	$7+\gamma 8$	2.4
1.2	$2+7-8$	2.3^{-1}
1.3	$7-8$	3.5
1.4	$6+8$	2.12
1.5	$4+7$	2.2^0
1.6	$4+6$	3.6
1.7	8	3.3
1.8	6	4.2
1.9	4	4.3
1.10	2	L_5

The constants v_0, ρ_0, and p_0 can be related in two ways:

(a) $$v_0 = 0, \qquad \frac{1-\omega}{\omega^4}\rho_0^2 + \Big(3 - \frac{2}{\omega(\gamma-1)} + 3\frac{1-\omega^2}{\omega^2}\frac{æ_0}{R}\Big)\rho_0 p_0 + 6\omega(\omega+1)\frac{æ_0}{R}p_0 = 0;$$

(b) $$\rho_0 = \frac{6\omega^3(2\omega+1)}{(2\omega+3)(\omega-1)}p_0, \quad v_0^2 = \frac{(1-\omega^2)(2\omega+3)}{2\omega^4}\frac{æ_0}{R} + \frac{2\omega^2-\omega-2}{2\omega^3} - \frac{2\omega+3}{3\omega^3}\frac{1}{\gamma-1}.$$

Submodel 3.5. Physical solutions exist only for $\omega \neq 0$ and $\omega \neq 1/2$:

$$u = \frac{2\omega-1}{2\omega}\frac{x}{t}, \quad v = v_0\frac{x}{t}, \quad \rho = \rho_0 x^{2(\omega-1)}t^{1-2\omega}, \quad p = p_0 x^{2\omega}t^{-1-2\omega}, \quad \Big(\frac{p_0}{\rho_0}\Big)^{\omega} = \frac{3\omega p_0}{2\omega-1} - \frac{3\rho_0}{8\omega^2}.$$

The constants v_0, ρ_0, and p_0 can be related in two ways:

(a) $$v_0 = 0, \quad \frac{12\omega(\omega+1)}{2\omega+1}\frac{\gamma-1}{R}æ_0 p_0^2 + \frac{-4\omega^2+4\omega-1}{8\omega^4}(\gamma-1)\rho_0^2 + \Big(\frac{1}{\omega} - \frac{3(\omega+1)}{2\omega^2}\frac{\gamma-1}{R}æ_0\Big)p_0\rho_0 = 0;$$

(b) $$v_0^2 = \frac{2\omega-1}{12\omega^3(2\omega+1)}\Big(3(2\omega-1)(2\omega+1)^2 + 2\frac{æ_0}{R}(2\omega+3)(\omega+1) + 4\omega^2 - 8\omega - 3\Big), \quad p_0 = \frac{4\omega^2+4\omega-3}{2\omega^3(2\omega+1)}\rho_0.$$

Submodel 3.6. Physical solutions exist only for $\omega = 0$, i.e., when μ and æ are constants:

$$u = u_0\sqrt{x}, \qquad v = t + v_0\sqrt{x}, \qquad \rho = \frac{v_0}{2(u_0v_0+2)x\sqrt{x}}, \qquad p = \frac{(5v_0+8)u_0}{2(u_0v_0+2)\sqrt{x}},$$

$$\Big(5\frac{\gamma-1}{R}æ_0 - \frac{11}{4}\gamma - \frac{21}{4}\Big)u_0^3v_0^2 + 2\Big(9\frac{\gamma-1}{R}æ_0 - 2\gamma\Big)u_0^2v_0$$
$$+ \frac{\gamma-1}{R}u_0v_0^4 + \frac{\gamma-1}{2}v_0^3 + 16\frac{\gamma-1}{R}æ_0u_0 = 0, \qquad u_0v_0 + 2 \neq 0.$$

Submodels 3.7 *and* 3.9 have no physical solutions.

Partially invariant solutions are a generalization of invariant solutions. In their construction an overdetermined system of differential equations arises which requires a compatibility study, for example, by an algorithm in [7]. This algorithm, providing an answer to the question of integrability of the system, does not necessarily lead to a visible representation of the solution. We give herein only two partially invariant solutions, which are reduced to invariant solutions and have the simplest form.

Submodel 3.10. The invariants of the subgroup are t, u/x, $\rho x^{2(1-\omega)}$, and $px^{-2\omega}$. We seek a solution in the form

$$u = xu_1(t), \qquad \rho = x^{2(\omega-1)}\rho_1(t), \qquad p = x^{2\omega}p_1(t).$$

After compatibility analysis of system (1.1)–(1.4) for v, we obtain $v = y/t + x\varphi(t)$. The functions $u_1(t)$, $\varphi(t)$, $\rho_1(t)$, and $p_1(t)$ are found from the equations

$$\rho_1' + 2\omega u_1\rho_1 + \rho_1/t = 0, \qquad \rho_1\Big(u_1' + u_1^2\Big) = -2\omega p_1 + \frac{4}{3}\omega\Big(u_1 - \frac{1}{t}\Big)\Big(\frac{p_1}{\rho_1}\Big)^{\omega},$$

$$\varphi' + \Big(u_1 + \frac{1}{t} - \frac{2\omega+1}{\rho_1}\Big(\frac{p_1}{\rho_1}\Big)^{\omega}\Big)\varphi = 0,$$

$$-p_1' + 4(\omega+1)\frac{\gamma-1}{R}æ_0\Big(\frac{p_1}{\rho_1}\Big)^{\omega+1} - 2(\omega+1)u_1p_1 - 2(\gamma-1)u_1p_1 - \gamma\frac{p_1}{t} + \frac{4}{3}(\gamma-1)\Big(u_1 - \frac{1}{t}\Big)^2\Big(\frac{p_1}{\rho_1}\Big)^{\omega} + \varphi^2\Big(\frac{p_1}{\rho_1}\Big)^{\omega} = 0.$$

Since the equations are solved with respect to the first derivative of all sought-for functions, the solution is invariant under the theorem of reduction [2, Theorem 22.7].

Submodel 3.11. Invariants of the subgroup: x, u, ρ, and p. We seek a solution in the form

$$u = u(x), \qquad \rho = \rho(x), \qquad p = p(x).$$

After studying the system of equations (1.1)–(1.4) for compatibility we obtain $u = c_1/x\rho$ and $v = v(x)$.

Let us consider subalgebra in 2.16. It is a subalgebra in 3.11 The invariant solution of submodel 2.16 has the form

$$u = u(x), \qquad v = v(x), \qquad \rho = \rho(x), \qquad p = p(x),$$

i.e., submodel 3.11 is a particular case of submodel 2.16 (a more careful analysis shows that these two models completely coincide). Since in the transition from submodel 3.11 to submodel 2.16 the rank of the solution is preserved and the defect decreases by one, the partially invariant solution 3.11 is reduced to the invariant solution 2.16.

Depending on the value of c_1, the solution can be described in two ways:
(a) $c_1 \neq 0$; the functions $v(x)$, $\rho(x)$, and $p(x)$ satisfy the equations

$$v'' = v'\Big(\omega\Big(\frac{\rho'}{\rho} - \frac{p'}{p}\Big) + \frac{1}{x}\Big(c_1\Big(\frac{\rho}{p}\Big)^{\omega} - 1\Big)\Big),$$

$$\rho'' = (\omega+2)\frac{\rho'^2}{\rho} - \omega\frac{\rho'p'}{p} + \Big(\frac{3}{4}c_1\Big(\frac{\rho}{p}\Big)^{\omega} + \frac{3}{2}\omega + 1\Big)\frac{\rho'}{\rho} - \frac{3}{4}\Big(\frac{\rho}{c_1x^2}\Big(\frac{\rho}{p}\Big)^{\omega} + \frac{2\omega}{p}\Big)\frac{p'}{x\rho} + \frac{3c_1\rho}{4x^2}\Big(\frac{\rho}{p}\Big)^{\omega},$$

$$p'' = -\frac{R}{æ_0}\rho v'^2 - \frac{4c_1^2R\rho'^2}{3æ_0x^2\rho^3} + (\omega+2)\frac{\rho'p'}{\rho} - \Big(c_1\frac{R}{æ_0}x^2\rho p\Big(\frac{\rho}{p}\Big)^{\omega}$$

$$+ 4c_1^2\frac{R}{æ_0} + \frac{c_1x^2\rho pR}{æ_0(\gamma-1)}\Big(\frac{\rho}{p}\Big)^{\omega} - \frac{3}{4}c_1x^2\rho p\Big(\frac{\rho}{p}\Big)^{\omega} - \frac{3\omega+4}{2}x^2\rho p\Big)\frac{\rho'}{x^3\rho^2}$$

$$-\omega\frac{p'^2}{p} - \Big(\frac{c_1R}{æ_0(\gamma-1)}\Big(\frac{\rho}{p}\Big)^{\omega} + \frac{3x^2\rho p}{4c_1}\Big(\frac{\rho}{p}\Big)^{\omega} + \frac{3}{2}\omega + 1\Big)\frac{p'}{x} - \Big(\frac{4c_1R}{æ_0\rho} - \frac{3}{4}x^2p\Big(\frac{\rho}{p}\Big)^{\omega}\Big)\frac{c_1}{x^4};$$

(b) $c_1 = 0$, $p \equiv p_0$; the functions $v(x)$, and $\rho(x)$ are restored from the equations

$$v' = c_2\rho^{\omega}/x, \qquad \rho'' + \Big(\frac{1}{x} - \frac{\omega+2}{\rho}\Big)\frac{\rho'}{\rho} - \frac{c_2\rho^{\omega-1}}{p_0x} = 0.$$

The remaining partially invariant solutions, which are constructed on the basis of three-dimensional subalgebras, have a cumbersome form and are not given here.

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